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Stochastic Reduced Basis Methods

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Stochastic reduced basis methods for solving large-scale linear random algebraic systems of equations, such as those obtained by discretizing linear stochastic partial differential equations in space, time, and the random dimension, are introduced. The fundamental idea employed is to represent the system response using a linear combination of stochastic basis vectors with undetermined deterministic coefficients (or random functions). We present a theoretical justification for employing basis vectors spanning the preconditioned stochastic Krylov subspace to approximate the response process. Subsequently, variants of the Bubnov–Galerkin scheme are employed to compute the undetermined coefficients, which allow explicit expressions for the response quantities to be derived. We also examine some theoretical properties of the projection scheme and procedures for computing the response statistics. Numerical studies are presented for static and dynamic analysis of stochastic structural systems. We demonstrate that significant improvements over the Neumann expansion scheme, as well as other relevant techniques in the literature, can be achieved.

I. Introduction

THE equations governing the physics of many complex systems can be described by ordinary or partial differential equations (PDEs). A wide body of numerical methods based on finite differences, finite elements, and boundary elements are available in the computational mechanics literature to solve the governing equations approximately for the response quantities of interest. Over the last 50 years, significant progress has been made in the theoretical groundwork of these methods for cases when a system is modeled in a deterministic framework and when a deterministic linear system is subjected to random excitation.

By contrast, the use of probabilistic models for the system parameters leads to a significant increase in the problem complexity. This is primarily due to the difficulties in arriving at tractable descriptions of the system response in terms of the stochastic differential operators and the random excitation fields. Exact solutions to this class of problems are possible only under restrictive assumptions for simple problems (for example, Elishakoff et al.[1]). In the context of stochastic analysis of large-scale systems of practical interest, most research work has focused on computationally efficient methods that allow the response statistics to be approximated with reasonable accuracy.

The approaches in the literature can be broadly classified into different categories, depending on how the system parameters are modeled (random field or random variables), the scheme used for discretizing the random fields, the linearization techniques employed to simplify the nonlinear terms, the spatial and temporal discretization scheme, and the algorithm used to solve the resulting random algebraic equations. The interested reader is referred to the monographs of Ghanem and Spanos[2] and Kleiber and Hien[3] for a detailed exposition of computational stochastic mechanics.

Monte Carlo simulation (MCS) techniques[4] and response surface methodology (RSM)[5] have been widely applied to a variety of problems in this area. These approaches are quite general in scope and utilize existing deterministic analysis software. However, due to the requirement of many deterministic simulations, they are practical only for problems where deterministic analysis takes modest computational effort. Local approximation methods based on Taylor or Neumann expansion series have also been popularly used in the stochastic finite element literature (for example, Refs. 3, 6, and 7). These methods are computationally more efficient compared to MCS and RSM. However, local approximation methods give accurate results only for small coefficients of variation of the random system parameters.

A spectral stochastic finite element method (SSFEM) was proposed by Ghanem and Spanos[2,8] for linear stochastic PDEs. In this approach, the random fields describing the PDE coefficients are discretized using the Karhunen–Loeve expansion scheme. Subsequently, a finite element procedure is used to derive a system of linear random algebraic equations, which is then approximately solved using the Neumann expansion scheme. An alternative approach was also proposed, wherein each component of the solution vector is represented by the polynomial chaos decomposition with unknown coefficients. By the use of the Bubnov–Galerkin scheme, a system of deterministic linear algebraic equations (with increased dimensionality proportional to the number of terms in the polynomial chaos) was obtained for the unknown coefficients. The reader is referred to Ghanem[9] for a recent review of the mathematical background and implementation aspects of the SSFEM.

More recently, Elishakoff et al.[10] and Ren and Elishakoff[11] have focused on finite element analysis of structures with large stochastic variations. The key idea has been to develop approaches that do not use perturbation schemes. An excellent discussion on the motivation for this and further work in the area of stochastic finite element analysis can be found in the work of Elishakoff and Ren.[12]

The present paper is concerned with the analysis of systems governed by linear stochastic PDEs. In particular, we introduce a class of efficient numerical schemes for solving large-scale linear random algebraic systems of equations, such as those obtained by discretizing linear stochastic PDEs in space, time, and the random dimension. The fundamental idea proposed here is to represent the response process using a linear combination of stochastic basis vectors with undetermined coefficients. The undetermined coefficients are either considered as deterministic scalars or random functions. Methods based on this representation, where the number of undetermined coefficients is less than the dimension of the discretized PDE, are referred to as stochastic reduced basis methods (SRBMs) in this paper. In contrast, in the projection scheme proposed by Ghanem and Spanos,[2] the basis vectors are considered to be unknown, which leads to a system of equations with increased dimensionality.

We discuss the choice of basis vectors, with a particular emphasis on the preconditioned stochastic Krylov subspace. A theoretical justification for employing the terms of the stochastic Krylov subspace as basis vectors is outlined. Efficient procedures for computing

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the basis vectors are presented, particularly for stochastic structural dynamic analysis. Subsequently, we employ two variants of the Bubnov–Galerkin scheme for computing the undetermined coefficients in the stochastic reduced basis representation. In the first procedure, the undetermined coefficients are considered as deterministic scalars. The second procedure deals with the case when only two or three basis vectors are used. Hence, it becomes possible to treat the reduced basis coefficients as random functions, which ensures that the stochastic residual error is orthogonal to the approximating space of basis vectors with probability one. Both of these projection schemes allow explicit expressions for the response quantities to be derived, which enables us to compute its complete probabilistic description efficiently. We also briefly discuss some theoretical properties of the stochastic Bubnov–Galerkin scheme.

Numerical studies on some problems in structural mechanics are presented to demonstrate that high-quality approximations of the response statistics can be achieved for moderate to large coefficients of variation of the random system parameters. In particular, it is shown that SRBMs can be up to orders of magnitude more accurate than the Neumann expansion scheme. Our results also suggest that SRBMs give results that are comparable to or better in accuracy than the polynomial chaos scheme at a much lower computational cost.

This paper is organized as follows: Section II presents some mathematical preliminaries used in the paper. In Sec. III, we present a theoretical justification for employing basis vectors spanning the preconditioned stochastic Krylov subspace. We also outline an efficient scheme for computing the basis vectors for stochastic structural dynamic analysis problems. Stochastic variants of the Bubnov–Galerkin scheme and its theoretical properties are presented in Sec. IV. In Sec. V, we examine the computational properties of SRBMs, including procedures for computing the response statistics. Numerical studies on static and dynamic analysis of stochastic structural systems are summarized in Sec. VI. Section VII concludes the paper and discusses some directions for further research.

II. Preliminaries

We use the following notation throughout this paper. For the sake of generality, all of the vector spaces considered are complex, unless otherwise stated. We use the notation $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$ to refer to the space of $n\times n$ real and complex matrices, respectively. Random quantities are indicated explicitly as a function of $\boldsymbol{\theta}$ or $\boldsymbol{\eta}$, and the ensemble average is denoted with angle brackets. We denote vectors and matrices by lower case and upper case bold characters, respectively. The notation $\boldsymbol{x}^*$ is used to denote the complex conjugate transpose of a vector or matrix (if it is complex), or the transpose (if it is real).

Next, we outline the steps involved in discretization of linear stochastic PDEs in space and the random dimension on the lines presented earlier by Ghanem.[9] The objective is to derive a general expression for the discretized finite dimensional random equations, which sets the stage for the development of SRBMs. For illustration, we consider a linear stochastic PDE of the form

$$\mathcal{T}_\alpha[u(\boldsymbol{x},t,\boldsymbol{\theta},\boldsymbol{\eta})]+\mathcal{L}_\beta[u(\boldsymbol{x},t,\boldsymbol{\theta},\boldsymbol{\eta})]=f(\boldsymbol{x},t,\boldsymbol{\eta}) \tag{1}$$

where $\boldsymbol{x}\in\mathbb{R}^2$ or $\mathbb{R}^3$ refers to the spatial coordinates, $u(\boldsymbol{x},t,\boldsymbol{\theta},\boldsymbol{\eta})$ denotes the field variable, $t\in\mathbb{R}^+$ refers to time, and $\boldsymbol{\theta},\boldsymbol{\eta}\in\Omega$ belongs to the Hilbert space of random variables. $\mathcal{T}_\alpha$ and $\mathcal{L}_\beta$ denote linear stochastic differential operators. These operators have coefficients $\alpha(\boldsymbol{x},\boldsymbol{\theta})$ and $\beta(\boldsymbol{x},\boldsymbol{\theta})$, which are considered to be second-order random fields. Here, $f(\boldsymbol{x},t,\boldsymbol{\eta})$ denotes the random excitation field for which a solution $u(\boldsymbol{x},t,\boldsymbol{\theta},\boldsymbol{\eta})$ is sought.

The random fields in Eq. (1) can be readily discretized to represent them in terms of a finite number of random variables using any of the techniques in the literature.[2,13] Let us consider, for the sake of simplicity, that the random fields appear as multiplicative terms in the differential operators. Hence, spatial discretization of the stochastic PDE and application of the appropriate boundary conditions lead to a matrix system of random differential equations. For example, for a linear structural system, we obtain a matrix system of second-order random differential equations of the form

$$\boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{u}}(t,\boldsymbol{\theta},\boldsymbol{\eta})+\boldsymbol{C}(\boldsymbol{\theta})\dot{\boldsymbol{u}}(t,\boldsymbol{\theta},\boldsymbol{\eta})+\boldsymbol{K}(\boldsymbol{\theta})\boldsymbol{u}(t,\boldsymbol{\theta},\boldsymbol{\eta})$$
$$=\boldsymbol{f}_o(t)+\sum_{i=1}^{q}\eta_i\boldsymbol{f}_i(t) \tag{2}$$

where

$$\boldsymbol{M}(\boldsymbol{\theta})=\boldsymbol{M}_o+\sum_{i=1}^{p}\theta_i\boldsymbol{M}_i\in\mathbb{R}^{n\times n}$$

$$\boldsymbol{K}(\boldsymbol{\theta})=\boldsymbol{K}_o+\sum_{i=1}^{p}\theta_i\boldsymbol{K}_i\in\mathbb{R}^{n\times n}$$

denote the random mass and stiffness matrix, respectively, where $\boldsymbol{M}_o$, $\boldsymbol{M}_i$, $\boldsymbol{K}_o$, and $\boldsymbol{K}_i\in\mathbb{R}^{n\times n}$ are deterministic matrices, and where n is the total number of degrees of freedom (DOF). $\boldsymbol{C}(\boldsymbol{\theta})\in\mathbb{R}^{n\times n}$ denotes the random damping matrix, which we will consider to be proportional [i.e., $\boldsymbol{C}(\boldsymbol{\theta})=\zeta_1\boldsymbol{K}(\boldsymbol{\theta})+\zeta_2\boldsymbol{M}(\boldsymbol{\theta})$], where ζ_1 and $\zeta_2\in\mathbb{R}$ are deterministic scalars. Here, $\boldsymbol{f}_o(t)$ and $\boldsymbol{f}_i(t)\in\mathbb{R}^n$ are deterministic vectors. Also, $\boldsymbol{\theta}=\{\theta_i\}$, $i=1,2,\ldots,p$ and $\boldsymbol{\eta}=\{\eta_i\}$, $i=1,2,\ldots,q$, denote the vectors of random variables arising from discretization of the random fields describing the system properties and the excitation field, respectively. Finally, $\boldsymbol{u}(t,\boldsymbol{\theta},\boldsymbol{\eta})\in\mathbb{R}^n$ denotes the discretized vector of displacements.

Clearly, for static problems, Eq. (2) will reduce to a system of linear random algebraic equations. Similar sets of equations could be arrived at for time-dependent problems by transforming the equations to the frequency domain, which leads to a system of complex linear random algebraic equations. As shown in a recent study,[14] a system of linear random algebraic equations can also be arrived at when the implicit Newmark scheme is used for computing the response in the time domain. Hence, without any loss of generality, we may represent the resulting system of linear random algebraic equations as

$$\left(\boldsymbol{L}+\sum_{i=1}^{p}\theta_i\boldsymbol{\Pi}_i\right)\boldsymbol{u}(\boldsymbol{\theta},\boldsymbol{\eta})=\boldsymbol{f}_o+\sum_{i=1}^{q}\eta_i\boldsymbol{f}_i \tag{3}$$

where $\boldsymbol{L}$ and $\boldsymbol{\Pi}_i\in\mathbb{R}^{n\times n}$ are deterministic matrices, $\boldsymbol{f}_o$ and $\boldsymbol{f}_i\in\mathbb{R}^n$ are deterministic vectors, and $\boldsymbol{u}(\boldsymbol{\theta},\boldsymbol{\eta})\in\mathbb{R}^n$ is the stochastic response process. In other words, a solution methodology for Eq. (3) can be directly applied to static and dynamic response analysis of linear stochastic structural systems, as well as a wider class of linear stochastic PDEs.

For cases when the system parameters and external forces are modeled as random variables, Eq. (3) can be derived by expanding the global coefficient matrices using a first-order Taylor series and appropriately representing the random forces. Note that this representation is exact when the coefficient matrix is a linear function of the random system parameters, for example, when the Young's modulus of a beam member is modeled as a random variable. Alternatively, the polynomial chaos decomposition scheme[2] can be employed to expand each term of the coefficient matrix and the force vector using a series of orthogonal random polynomials. Then the random variables θ_i and η_i become orthogonal random polynomials. Even though the formulations that follow assume a random variable representation, it can be readily extended to cases involving expansion in terms of random polynomials.

In the sections that follow, the coefficient matrix of Eq. (3) will sometimes be denoted by the matrix $\boldsymbol{A}(\boldsymbol{\theta})$. This implies that $\langle\boldsymbol{A}(\boldsymbol{\theta})\rangle=\boldsymbol{L}$, and the term

$$\sum_{i=1}^{p}\theta_i\boldsymbol{\Pi}_i$$

denotes the zero-mean random components of the matrix $\boldsymbol{A}(\boldsymbol{\theta})$. Similarly, the right-hand side (RHS) of Eq. (3) will sometimes be compactly written as $\boldsymbol{f}(\boldsymbol{\eta})$.

III. Stochastic Basis Vectors

In this section, we present the basic idea behind SRBMs. We briefly outline why the stochastic Krylov subspace is appropriate

for computing the solution of Eq. (3). Then we describe a recursive scheme for computing basis vectors spanning the preconditioned stochastic Krylov subspace. Particular attention is paid to efficient schemes for basis vectors computation in stochastic structural dynamics. To proceed, let us first compactly rewrite Eq. (3) as

$$\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{u}(\boldsymbol{\theta}, \boldsymbol{\eta}) = \boldsymbol{f}(\boldsymbol{\eta}) \tag{4}$$

where $\boldsymbol{A}(\boldsymbol{\theta}) \in \mathbb{C}^{n \times n}$ is a random matrix and $\boldsymbol{u}(\boldsymbol{\theta}, \boldsymbol{\eta})$ and $\boldsymbol{f}(\boldsymbol{\eta}) \in \mathbb{C}^n$ are random vectors. Also, $\boldsymbol{\theta} \in \mathbb{R}^p$ and $\boldsymbol{\eta} \in \mathbb{R}^q$ denote the vector of random variables in the coefficient matrix and the RHS, respectively. Note that, for the sake of generality, the coefficient matrix and the RHS are considered here as complex quantities. Henceforth, for simplicity of presentation, all random quantities (except for the RHS $\boldsymbol{f}$) will be denoted as a function of $\boldsymbol{\theta}$. The dependence on the random vector $\boldsymbol{\eta}$ will not be shown explicitly.

The fundamental idea used in this paper is to represent the solution of Eq. (4) as

$$\hat{\boldsymbol{u}}(\boldsymbol{\theta}) = \xi_0(\boldsymbol{\theta})\psi_0(\boldsymbol{\theta}) + \xi_1(\boldsymbol{\theta})\psi_1(\boldsymbol{\theta}) + \cdots + \xi_m(\boldsymbol{\theta})\psi_m(\boldsymbol{\theta}) = \boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi}(\boldsymbol{\theta}) \tag{5}$$

where $\boldsymbol{\Psi}(\boldsymbol{\theta}) = [\psi_0(\boldsymbol{\theta}), \psi_1(\boldsymbol{\theta}), \ldots, \psi_m(\boldsymbol{\theta})] \in \mathbb{C}^{n \times (m+1)}$ denotes a matrix of $m+1$ stochastic basis vectors and $\boldsymbol{\xi}(\boldsymbol{\theta}) = \{\xi_0(\boldsymbol{\theta}), \xi_1(\boldsymbol{\theta}), \ldots, \xi_m(\boldsymbol{\theta})\} \in \mathbb{C}^{m+1}$ denotes the vector of undetermined coefficients in the reduced basis. Note that for the sake of computational efficiency we will choose $m \lll n$.

Clearly, along with the number of terms, the accuracy of the stochastic reduced basis representation will depend on the quality of the basis vectors as well as the scheme employed to compute the undetermined coefficients in Eq. (5). We discuss next which basis vectors are appropriate for approximating the solution process $\boldsymbol{u}(\boldsymbol{\theta})$.

A. Properties of the Stochastic Krylov Subspace

In this section, we show that the solution of Eq. (4) lies in the stochastic Krylov subspace. The analysis presented in this section follows from the theoretical results outlined by Ipsen and Meyer[15] for deterministic systems of equations. More specifically, we consider the problem where, given a nonsingular random square matrix $\boldsymbol{A}(\boldsymbol{\theta})$, it is required to compute the vector $\boldsymbol{A}(\boldsymbol{\theta})^{-1}\boldsymbol{f}(\boldsymbol{\eta})$. Let us first introduce the notion of the minimal polynomial of a random matrix. For a random matrix, the minimal polynomial q can be defined as the unique monic random polynomial of smallest degree such that $q[\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{\theta}] = 0$. We assume that the matrix $\boldsymbol{A}(\boldsymbol{\theta})$ is diagonalizable and that its d distinct eigenvalues can be written as $\lambda_1(\boldsymbol{\theta}), \lambda_2(\boldsymbol{\theta}), \ldots, \lambda_d(\boldsymbol{\theta})$. Then, it follows that

$$q[\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{\theta}] = \prod_{j=1}^{d} [\boldsymbol{A}(\boldsymbol{\theta}) - \lambda_j(\boldsymbol{\theta})\boldsymbol{I}]^{m_j} \tag{6}$$

where m_j denotes the index of the jth eigenvalue, and

$$m = \sum_{j=1}^{d} m_j$$

From Eq. (6), the minimal polynomial of $\boldsymbol{A}(\boldsymbol{\theta})$ can be written as

$$q[\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{\theta}] = \sum_{j=0}^{m} \gamma_j(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\theta})^j = 0 \tag{7}$$

where the term

$$\gamma_0(\boldsymbol{\theta}) = \prod_{j=1}^{d} [-\lambda_j(\boldsymbol{\theta})]^{m_j} \neq 0$$

for any realization of $\boldsymbol{\theta}$ because we have assumed that the random matrix $\boldsymbol{A}(\boldsymbol{\theta})$ is nonsingular. Hence, the inverse of $\boldsymbol{A}(\boldsymbol{\theta})$ can be written as

$$\boldsymbol{A}(\boldsymbol{\theta})^{-1} = -\frac{1}{\gamma_0(\boldsymbol{\theta})} \sum_{j=0}^{m-1} \gamma_{j+1}(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\theta})^j \tag{8}$$

It can be clearly seen from the preceding equation that the inverse of a nonsingular random matrix lies in the space spanned by the terms of the minimal random polynomial. This implies that the vector $\boldsymbol{A}(\boldsymbol{\theta})^{-1}\boldsymbol{f}(\boldsymbol{\eta})$ belongs to the stochastic Krylov subspace defined hereafter.

Definition 1: The stochastic Krylov subspace of order m is defined as

$$\mathcal{K}_m[\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{f}(\boldsymbol{\eta})] = \text{span}\{\boldsymbol{f}(\boldsymbol{\eta}), \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{f}(\boldsymbol{\eta}), \boldsymbol{A}(\boldsymbol{\theta})^2\boldsymbol{f}(\boldsymbol{\eta}), \ldots, \boldsymbol{A}(\boldsymbol{\theta})^{m-1}\boldsymbol{f}(\boldsymbol{\eta})\}$$

where $\boldsymbol{A}(\boldsymbol{\theta}) \in \mathbb{C}^{n \times n}$ is a random matrix, and $\boldsymbol{f}(\boldsymbol{\eta}) \in \mathbb{C}^n$ is either a deterministic or a random vector.

The dimension of the Krylov subspace required to compute high-quality approximations will depend on the degree of the minimal polynomial of the random matrix. This observation can be formally stated as follows.

Theorem 1: If the minimal random polynomial of a nonsingular random square matrix $\boldsymbol{A}(\boldsymbol{\theta})$ has degree m, then the solution to $\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{u}(\boldsymbol{\theta}) = \boldsymbol{f}(\boldsymbol{\eta})$ lies in the stochastic Krylov subspace $\mathcal{K}_m[\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{f}(\boldsymbol{\eta})]$.

Hence, in principle, $\mathcal{K}_m[\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{f}(\boldsymbol{\eta})]$ provides a stochastic subspace, by the use of which an approximation for the random vector $\boldsymbol{A}(\boldsymbol{\theta})^{-1}\boldsymbol{f}(\boldsymbol{\eta})$ can be computed. However, Theorem 1 implies that in practice a large number of basis vectors would be required to compute accurate results because for many cases we expect the degree of the minimal polynomial m to be close to n. To achieve accurate results using few basis vectors, we need to transform the coefficient matrix $\boldsymbol{A}(\boldsymbol{\theta})$ such that the probability density functions (PDFs) of its eigenvalues show a high degree of overlap. Thus, from a numerical viewpoint, the degree of the minimal polynomial of the transformed coefficient matrix will be much smaller compared to n. We show next how this can be achieved by employing a preconditioning scheme.

B. Preconditioned Stochastic Krylov Subspace

Based on the preceding discussion, a straightforward choice of basis vectors would be the mth-order stochastic Krylov subspace $\mathcal{K}_m[\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{f}(\boldsymbol{\eta})]$. This suggests that the stochastic basis vectors can be computed as

$$\psi_0 = \boldsymbol{f}(\boldsymbol{\eta}), \qquad \psi_1 = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{f}(\boldsymbol{\eta}), \ldots, \psi_{m-1} = \boldsymbol{A}(\boldsymbol{\theta})^{m-1}\boldsymbol{f}(\boldsymbol{\eta}) \tag{9}$$

However, as mentioned earlier, the number of stochastic basis vectors required to compute accurate approximations of the solution vector could be as high as n. For the sake of computational efficiency, it is desirable to use only a small number of basis vectors. In order to arrive at a richer stochastic subspace, we employ a preconditioning approach to transform the coefficient matrix. In the present study, we choose the deterministic matrix $\langle \boldsymbol{A}(\boldsymbol{\theta}) \rangle^{-1} = \boldsymbol{L}^{-1}$ as the preconditioner. (A good choice of preconditioning matrix $\boldsymbol{M}$ is one for which $\boldsymbol{MA}$ is close to an identity matrix or a matrix with highly clustered eigenvalues.) This choice of the preconditioning matrix would reduce the degree of the minimal polynomial of the transformed coefficient matrix. In other words, because the eigenvalues of $\boldsymbol{L}^{-1}\boldsymbol{A}(\boldsymbol{\theta})$ will be clustered around unity for small stochasticity, the PDFs of its eigenvalues will have a high degree of overlap. This, in turn, would allow us to compute high-quality approximations using a small number of basis vectors.

The left preconditioned version of Eq. (4) can be written as

$$\boldsymbol{L}^{-1}\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{u}(\boldsymbol{\theta}) = \boldsymbol{L}^{-1}\boldsymbol{f}(\boldsymbol{\eta}) \tag{10}$$

Note that the solutions of Eqs. (10) and (4) are the same. Before proceeding further, we shall briefly illustrate the relationship between the left preconditioned stochastic Krylov subspace (when $\boldsymbol{L}^{-1}$ is used as a preconditioner) and the Neumann series given hereafter,

$$\tilde{\boldsymbol{u}}(\boldsymbol{\theta}) = \sum_{i=0}^{\infty} (-1)^i \left(\boldsymbol{L}^{-1} \sum_{i=1}^{p} \theta_i \boldsymbol{\Pi}_i \right)^i \boldsymbol{L}^{-1} \left(\boldsymbol{f}_o + \sum_{i=1}^{q} \eta_i \boldsymbol{f}_i \right) \tag{11}$$

By the use of the analysis presented in a recent study,[16] it can shown that for this particular choice of the preconditioner, the terms of the

Neumann series also span the left preconditioned stochastic Krylov subspace, that is,

$$\mathcal{K}_m[\boldsymbol{L}^{-1}\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{L}^{-1}\boldsymbol{f}(\boldsymbol{\eta})] = \mathcal{K}_m\left[\boldsymbol{L}^{-1}\sum_{i=1}^{p}\theta_i\boldsymbol{\Pi}_i, \boldsymbol{L}^{-1}\boldsymbol{f}(\boldsymbol{\eta})\right] \quad (12)$$

The implication of the preceding result is that using the preconditioner $\boldsymbol{L}^{-1}$ is equivalent to using the terms of the Neumann series as stochastic basis vectors. It can be readily seen that each basis vector in the preconditioned stochastic Krylov subspace (except for the first one) can be written as a vector of homogeneous random polynomials. To simplify our notation, we shall now adopt the Einstein convention that whenever the same index appears twice in an expression, summation with respect to that index over its range is implied. The first three basis vectors can be explicitly written as a function of $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ as

$$\boldsymbol{\psi}_0(\boldsymbol{\theta}) = \boldsymbol{L}^{-1}\boldsymbol{f}_o + \eta_l\boldsymbol{L}^{-1}\boldsymbol{f}_i \quad (13)$$

$$\boldsymbol{\psi}_1(\boldsymbol{\theta}) = \theta_i\left(\boldsymbol{b}_i^1 + \eta_j\boldsymbol{c}_{ij}^1\right) \quad (14)$$

$$\boldsymbol{\psi}_2(\boldsymbol{\theta}) = \theta_{i_1}\theta_{i_2}\left(\boldsymbol{b}_{i_1i_2}^2 + \eta_{i_3}\boldsymbol{c}_{i_1i_2i_3}^2\right) \quad (15)$$

where $\boldsymbol{b}_i^1 = \boldsymbol{L}^{-1}\boldsymbol{\Pi}_i\boldsymbol{L}^{-1}\boldsymbol{f}_o$, $\boldsymbol{c}_{ij}^1 = \boldsymbol{L}^{-1}\boldsymbol{\Pi}_i\boldsymbol{L}^{-1}\boldsymbol{f}_j$, $\boldsymbol{b}_{i_1i_2}^2 = \boldsymbol{L}^{-1}\boldsymbol{\Pi}_{i_1}\boldsymbol{L}^{-1}\boldsymbol{\Pi}_{i_2}\boldsymbol{L}^{-1}\boldsymbol{f}_o$, and $\boldsymbol{c}_{i_1i_2i_3}^2 = \boldsymbol{L}^{-1}\boldsymbol{\Pi}_{i_1}\boldsymbol{L}^{-1}\boldsymbol{\Pi}_{i_2}\boldsymbol{L}^{-1}\boldsymbol{f}_{i_3} \in \mathbb{C}^n$ are deterministic vectors.

A general expression for the kth stochastic basis vector can be written as

$$\boldsymbol{\psi}_k(\boldsymbol{\theta}) = \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_k}\left(\boldsymbol{b}_{i_1i_2,\ldots,i_k}^k + \eta_{i_{k+1}}\boldsymbol{c}_{i_1i_2,\ldots,i_{k+1}}^k\right) \quad (16)$$

where $\boldsymbol{b}_{i_1i_2,\ldots,i_k}^k = \boldsymbol{L}^{-1}\boldsymbol{\Pi}_{i_1}\boldsymbol{L}^{-1}\boldsymbol{\Pi}_{i_2},\ldots,\boldsymbol{L}^{-1}\boldsymbol{\Pi}_{i_k}\boldsymbol{L}^{-1}\boldsymbol{f}_o$ and $\boldsymbol{c}_{i_1i_2,\ldots,i_{k+1}}^k = \boldsymbol{L}^{-1}\boldsymbol{\Pi}_{i_1}\boldsymbol{L}^{-1}\boldsymbol{\Pi}_{i_2},\ldots,\boldsymbol{L}^{-1}\boldsymbol{\Pi}_{i_k}\boldsymbol{L}^{-1}\boldsymbol{f}_{i_{k+1}} \in \mathbb{C}^n$ are deterministic vectors.

Because the matrices $\boldsymbol{L}$ and $\boldsymbol{\Pi}_i$ are expected to be highly sparse, the deterministic tensors $\boldsymbol{b}^k$ and $\boldsymbol{c}^k$ can be efficiently computed in a recursive fashion by solving the deterministic system of equations

$$\boldsymbol{L}\boldsymbol{b}_{i_1i_2,\ldots,i_{k+1}}^{k+1} = \boldsymbol{\Pi}_{i_{k+1}}\boldsymbol{b}_{i_1i_2,\ldots,i_k}^k, \qquad \boldsymbol{L}\boldsymbol{c}_{i_1i_2,\ldots,i_{k+2}}^{k+1} = \Pi_{i_{k+2}}\boldsymbol{c}_{i_1i_2,\ldots,i_{k+1}}^k \quad (17)$$

If the decomposed form of the matrix $\boldsymbol{L}$ is available, then Eq. (17) can be readily solved using forward and backward substitutions. However, note that computation of the higher-order basis vectors spanning the preconditioned stochastic Krylov subspace inevitably leads to an exponential increase in memory requirements, which could potentially become prohibitive for systems with a large number of DOF and random variables. One way to reduce the memory requirements would be to neglect the interaction terms in the basis vectors of order greater than two. For example, the fourth basis vector may be rewritten as

$$\boldsymbol{\psi}_3(\boldsymbol{\theta}) = \theta_i^3\left(\boldsymbol{b}_i^3 + \eta_j\boldsymbol{c}_{ij}^3\right) \quad (18)$$

where $\boldsymbol{b}_i^3 = \boldsymbol{L}^{-1}\boldsymbol{\Pi}_i\boldsymbol{L}^{-1}\boldsymbol{\Pi}_i\boldsymbol{L}^{-1}\boldsymbol{\Pi}_i\boldsymbol{L}^{-1}\boldsymbol{f}_o$ and $\boldsymbol{c}_{ij}^3 = \boldsymbol{L}^{-1}\boldsymbol{\Pi}_i\boldsymbol{L}^{-1}\boldsymbol{\Pi}_i\boldsymbol{L}^{-1}\boldsymbol{\Pi}_i\boldsymbol{L}^{-1}\boldsymbol{f}_j \in \mathbb{C}^n$ are deterministic vectors. By the use of the simplified basis vectors of the form shown in Eq. (18), the memory requirements and the computational complexity can be significantly reduced. However, in the numerical studies presented here, we exactly compute the terms of the preconditioned stochastic Krylov subspace without resorting to any simplification.

C. Basis Vectors for Stochastic Structural Dynamics

The procedure described earlier cannot be efficiently applied to compute the basis vectors for stochastic structural dynamic analysis problems. This is primarily because a system of random equations must be solved at each excitation frequency of interest, or at each time step. For the sake of simplicity, we shall consider here the case involving frequency-domain analysis of a proportionally damped stochastic structural system. The procedure presented can be readily extended to time-domain analysis.

For frequency-domain analysis, the deterministic and random components of the coefficient matrix $\boldsymbol{A}(\boldsymbol{\theta})$ can be defined as

$$\boldsymbol{L}(\omega) = (1 + j\zeta_1\omega)\boldsymbol{K}_o - \left(\omega^2 - j\zeta_2\omega\right)\boldsymbol{M}_o \quad (19)$$

$$\boldsymbol{\Pi}_i(\omega) = (1 + j\zeta_1\omega)\boldsymbol{K}_i - \left(\omega^2 - j\zeta_2\omega\right)\boldsymbol{M}_i \quad (20)$$

where ω is the excitation frequency of interest, ζ_1 and ζ_2 are the proportional damping coefficients defined earlier in Sec. II, and $j = \sqrt{-1}$. Note that the other matrices appearing in Eqs. (19) and (20) have been defined earlier in Sec. II.

If the stochastic basis vectors defined earlier are to be computed at each excitation frequency of interest ω, then the complex matrix $\boldsymbol{L}(\omega)$ has to be repeatedly inverted. This is because an independent set of stochastic basis vectors are used to compute the statistics of the response at each excitation frequency. This may lead to a significant increase in the computational cost if the response at a large number of frequency points is to be computed. To reduce the computational cost, we first solve for the left and right eigenvectors corresponding to the k lowest eigenvalues of the generalized deterministic eigenvalue problem $\boldsymbol{K}_o\phi = \lambda\boldsymbol{M}_o\phi$. Let $\Lambda \in \mathbb{C}^{k\times k}$ denote the diagonal matrix of eigenvalues and Φ_1 and $\Phi_2 \in \mathbb{C}^{n\times k}$ denote the matrix of left and right eigenvectors, respectively.

From Eq. (17), it can be observed that the tensors used in the stochastic basis vector representation can be computed by solving a deterministic system of equations of the form

$$\boldsymbol{L}(\omega)\boldsymbol{x}_1(\omega) = \boldsymbol{\Pi}_i(\omega)\boldsymbol{x}_2(\omega) \quad (21)$$

For efficiency reasons, we approximately solve Eq. (21) using the modal transformation $\boldsymbol{x}_1(\omega) = \Phi_2\boldsymbol{q}(\omega)$, where $\boldsymbol{q}(\omega) \in \mathbb{C}^k$. Note that this approximation was originally suggested by Nair and Keane[17] in the context of dynamic reanalysis. Hence, by the use of the property of biorthogonality of the eigenvectors, the solution of Eq. (21) can be approximated as

$$\boldsymbol{x}_1(\omega) = \Phi_2\left[\Lambda - \omega^2\boldsymbol{I}_k + j\omega(\zeta_1\Lambda + \zeta_2\boldsymbol{I}_k)\right]^{-1}\Phi_1^*\boldsymbol{\Pi}_i(\omega)\boldsymbol{x}_2(\omega) \quad (22)$$

where $\boldsymbol{I}_k \in \mathbb{R}^{k\times k}$ is the identity matrix. Because the matrix to be inverted in Eq. (22) is diagonal, the basis vectors can be efficiently computed. Furthermore, the accuracy of the approximation can be improved by increasing k or by employing a modal acceleration scheme.

D. Remarks

From Eqs. (16) and (5), it can be seen that the final solution turns out to be a vector of random polynomials. In particular, when the basis vectors are computed by expanding the coefficient matrix using a Taylor series, the basis vectors can be interpreted as the terms of the Taylor series for the solution vector. Note that this connection arises only if the matrix $\boldsymbol{L}^{-1}$ is used as the preconditioner. It is tempting here to draw connections between SRBMs and perturbation methods. However, there is a fundamental difference between these two classes of methods. First, perturbation methods have a limited radius of convergence due to which they may fail to converge when the coefficients of variation of θ_i are large. In contrast, SRBMs represent a physics-based stochastic analysis approach because the undetermined coefficients in Eq. (5) are computed by ensuring that the governing algebraic equation (4) is satisfied in some sense; see Sec. IV. Furthermore, due to Theorem 1, SRBMs can theoretically convergence to the exact solution, provided a sufficient number of basis vectors is used. Also note that the terms of the Neumann series were earlier used with a great degree of success as basis vectors for static structural reanalysis by Kirsch.[18] We discuss next the implementation aspects of the Bubnov–Galerkin scheme for stochastic problems and outline some of its theoretical properties.

IV. Stochastic Subspace Projection

In this section, we present variants of the Bubnov–Galerkin (BG) scheme for computing the undetermined coefficients in the stochastic reduced basis representation. The first step in the BG scheme is to formulate a stochastic residual error vector of the form

$$\boldsymbol{r}(\boldsymbol{\theta}) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi}(\boldsymbol{\theta}) - \boldsymbol{f}(\boldsymbol{\eta}) \quad (23)$$

The BG scheme is an orthogonal projection technique, which enforces the condition $\boldsymbol{r}(\boldsymbol{\theta}) \perp \boldsymbol{\Psi}(\boldsymbol{\theta})$. To implement this condition, we consider the following definitions for orthogonality between two random vectors.

Definition 2: Two random vectors $\boldsymbol{x}_1(\boldsymbol{\theta})$ and $\boldsymbol{x}_2(\boldsymbol{\theta}) \in \mathbb{C}^n$ are orthogonal in the Hilbert space of random variables if $\langle \boldsymbol{x}_1^*(\boldsymbol{\theta}) \boldsymbol{x}_2(\boldsymbol{\theta}) \rangle = 0$.

Definition 3: Two random vectors $\boldsymbol{x}_1(\boldsymbol{\theta})$ and $\boldsymbol{x}_2(\boldsymbol{\theta}) \in \mathbb{C}^n$ are orthogonal with probability one if $P[\boldsymbol{x}_1^*(\boldsymbol{\theta}) \boldsymbol{x}_2(\boldsymbol{\theta}) = 0] = 1$.

The reader is referred to Ghanem and Spanos[2] for a more formal review of the origins of definition 2. Definition 3 is a probabilistic interpretation of the definition of orthogonality between two deterministic vectors, which suggests that $\boldsymbol{\xi}(\boldsymbol{\theta})$ must be computed by solving the reduced-order system random algebraic system of equations

$$\tilde{\boldsymbol{A}}(\boldsymbol{\theta}) \boldsymbol{\xi}(\boldsymbol{\theta}) = \tilde{\boldsymbol{f}}(\boldsymbol{\theta}, \boldsymbol{\eta}) \tag{24}$$

where $\tilde{\boldsymbol{A}}(\boldsymbol{\theta}) = \boldsymbol{\Psi}^*(\boldsymbol{\theta}) \boldsymbol{A}(\boldsymbol{\theta}) \boldsymbol{\Psi}(\boldsymbol{\theta}) \in \mathbb{C}^{(m+1)\times(m+1)}$ and $\tilde{\boldsymbol{f}}(\boldsymbol{\theta}, \boldsymbol{\eta}) = \boldsymbol{\Psi}^*(\boldsymbol{\theta}) \boldsymbol{f}(\boldsymbol{\eta}) \in \mathbb{C}^{m+1}$ are the reduced random coefficient matrix and RHS, respectively. As shown later, the random matrix $\tilde{\boldsymbol{A}}(\boldsymbol{\theta})$ must be symbolically inverted to ensure that for each realization of $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ Eq. (24) holds. This is readily possible only when two or three basis vectors are used. A more practical approach to this problem is to employ simulation schemes for computing the statistics of the undetermined coefficients. However, we will not pursue this further here.

Now consider the case when definition 2 is employed to enforce the condition $\boldsymbol{r}(\boldsymbol{\theta}) \perp \boldsymbol{\Psi}(\boldsymbol{\theta})$. Then, the undetermined coefficients turn out to be deterministic scalars, which are computed by solving the ensemble averaged linear algebraic system of equations

$$\langle \tilde{\boldsymbol{A}}(\boldsymbol{\theta}) \rangle \boldsymbol{\xi} = \langle \tilde{\boldsymbol{f}}(\boldsymbol{\theta}, \boldsymbol{\eta}) \rangle \tag{25}$$

It can be readily shown that Eq. (25) is a zero-order approximation of Eq. (24). To show this, let us first approximate the solution of Eq. (24) using the first term of the Neumann series, that is, a zero-order approximation, which gives

$$\langle \tilde{\boldsymbol{A}}(\boldsymbol{\theta}) \rangle \hat{\boldsymbol{\xi}} = \tilde{\boldsymbol{f}}(\boldsymbol{\theta}, \boldsymbol{\eta}) \tag{26}$$

Clearly, if the RHS of Eq. (26) is replaced by its ensemble average, then the preceding equation becomes equivalent to Eq. (25). Hence, we refer to Eqs. (25) and (24) as the zero-order and the exact BG scheme, respectively. We present next implementation details of both of these variants of the BG scheme.

A. Zero-Order BG Scheme

The undetermined coefficients in the reduced basis are evaluated here such that the stochastic residual error $\boldsymbol{r}(\boldsymbol{\theta})$ is orthogonal to $\boldsymbol{\Psi}(\boldsymbol{\theta})$ in the sense of definition 2, that is,

$$\langle \boldsymbol{\Psi}^*(\boldsymbol{\theta}) \boldsymbol{r}(\boldsymbol{\theta}) \rangle = 0 \tag{27}$$

Equation (27) leads to a deterministic matrix system of equations of dimension $(m+1)$ for the coefficient vector $\boldsymbol{\xi}$. This formulation is henceforth referred to as SRBM-BG$_0$ to indicate that the zero-order BG scheme is used for stochastic subspace projection. The order of the approximation is considered to be equal to m, when $m+1$ basis vectors are used. The system of equations to be solved for the coefficient vector $\boldsymbol{\xi} \in \mathbb{C}^{m+1}$ can be written as

$$\left[\langle \boldsymbol{\Psi}^*(\boldsymbol{\theta}) \boldsymbol{L} \boldsymbol{\Psi}(\boldsymbol{\theta}) \rangle + \sum_{i=1}^{p} \langle \theta_i \boldsymbol{\Psi}^*(\boldsymbol{\theta}) \boldsymbol{\Pi}_i \boldsymbol{\Psi}(\boldsymbol{\theta}) \rangle \right] \boldsymbol{\xi} = \left\langle \boldsymbol{\Psi}^*(\boldsymbol{\theta}) \left(\boldsymbol{f}_o + \sum_{i=1}^{q} \eta_i \boldsymbol{f}_i \right) \right\rangle \tag{28}$$

The preceding equation can be rewritten in a compact form as

$$[\langle \boldsymbol{L}_R(\boldsymbol{\theta}) \rangle + \langle \boldsymbol{\Pi}_R(\boldsymbol{\theta}) \rangle] \boldsymbol{\xi} = \langle \tilde{\boldsymbol{f}}(\boldsymbol{\theta}, \boldsymbol{\eta}) \rangle \tag{29}$$

where $\langle \boldsymbol{L}_R(\boldsymbol{\theta}) \rangle$ and $\langle \boldsymbol{\Pi}_R(\boldsymbol{\theta}) \rangle \in \mathbb{C}^{(m+1)\times(m+1)}$ are deterministic matrices and $\langle \tilde{\boldsymbol{f}}(\boldsymbol{\theta}, \boldsymbol{\eta}) \rangle \in \mathbb{C}^{m+1}$ is the deterministic reduced RHS. These ensemble-averaged reduced-order terms can be readily computed using the joint statistics of $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$. General expressions for the elements of these terms are presented in Appendix A. Note that these expressions can be simplified when $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are jointly Gaussian, or when the interaction terms in the basis vectors of order greater than two are assumed to be zero as discussed earlier in Sec. III.B. These allow the computation of the deterministic coefficient vector $\boldsymbol{\xi}$, which we may then use in conjunction with Eq. (5) to arrive at an explicit expression for $\hat{\boldsymbol{u}}(\boldsymbol{\theta})$.

B. Exact BG Scheme

In contrast to the earlier formulation, here we aim to enforce the condition that the stochastic residual error vector $\boldsymbol{r}(\boldsymbol{\theta})$ is orthogonal to the approximating space $\boldsymbol{\Psi}(\boldsymbol{\theta})$ with probability one. In the sense of definition 3, this condition can be written as

$$P[\boldsymbol{\Psi}^*(\boldsymbol{\theta}) \boldsymbol{r}(\boldsymbol{\theta}) = 0] = 1 \tag{30}$$

It can be seen that Eq. (30) will be satisfied only when $\boldsymbol{\xi}(\boldsymbol{\theta})$ is computed as

$$\boldsymbol{\xi}(\boldsymbol{\theta}) = \left[\boldsymbol{\Psi}^*(\boldsymbol{\theta}) \left(\boldsymbol{L} + \sum_{i=1}^{p} \theta_i \boldsymbol{\Pi}_i \right) \boldsymbol{\Psi}(\boldsymbol{\theta}) \right]^{-1} \times \boldsymbol{\Psi}^*(\boldsymbol{\theta}) \left(\boldsymbol{f}_o + \sum_{i=1}^{q} \eta_i \boldsymbol{f}_i \right) \tag{31}$$

Hence, to achieve an explicit expression for $\boldsymbol{\xi}(\boldsymbol{\theta})$, we will have to invert symbolically a random matrix of dimension $(m+1)$. This can be readily carried out for small values of m. Our earlier experience suggests that generally two or three basis vectors are sufficient to ensure good approximations for moderate coefficients of variation of θ_i. Consider, for example, the first-order SRBM where $\boldsymbol{u}(\boldsymbol{\theta})$ is approximated using two basis vectors as shown:

$$\hat{\boldsymbol{u}}(\boldsymbol{\theta}) = \xi_0(\boldsymbol{\theta}) \left(\boldsymbol{L}^{-1} \boldsymbol{f}_o + \sum_{i=1}^{q} \eta_i \boldsymbol{L}^{-1} \boldsymbol{f}_i \right) + \xi_1(\boldsymbol{\theta}) \times \sum_{i=1}^{p} \theta_i \left(\boldsymbol{b}_i^1 + \sum_{j=1}^{q} \eta_j \boldsymbol{c}_{ij}^1 \right) \tag{32}$$

The corresponding 2×2 matrix system of equations to be solved for $\boldsymbol{\xi}(\boldsymbol{\theta})$ can be written as

$$[\boldsymbol{L}_R(\boldsymbol{\theta}) + \boldsymbol{\Pi}_R(\boldsymbol{\theta})] \boldsymbol{\xi}(\boldsymbol{\theta}) = \tilde{\boldsymbol{f}}(\boldsymbol{\theta}, \boldsymbol{\eta}) \tag{33}$$

Explicit expressions for $\boldsymbol{L}_R(\boldsymbol{\theta})$, $\boldsymbol{\Pi}_R(\boldsymbol{\theta})$, and $\tilde{\boldsymbol{f}}(\boldsymbol{\theta}, \boldsymbol{\eta})$ can be readily derived in terms of the random variables (see Appendix B). These explicit expressions can be used to compute the random functions $\xi_0(\boldsymbol{\theta})$ and $\xi_1(\boldsymbol{\theta})$. Similar equations can also be derived for the case when three basis vectors are used. The use of symbolic computation software is expected to alleviate the tediousness of the derivation greatly. Furthermore, it is expected that the explicit expressions for the reduced-order random matrices can be simplified by replacing terms of order greater than three by their ensemble averages without significant loss of accuracy. In summary, the exact BG scheme leads to an explicit but complicated expression for the response process. The implication of this on the computational procedure for calculating the response statistics is discussed later in Sec. V.

C. Theoretical Properties

The zero-order BG scheme has a number of interesting properties because it is an orthogonal projection scheme. For example, consider the case when the coefficient matrix $\boldsymbol{A}(\boldsymbol{\theta})$ is Hermitian positive definite. Furthermore, let $\mathcal{K}_m$ denote the stochastic subspace spanned by the set of orthogonal basis vectors $\psi_1(\boldsymbol{\theta}), \psi_2(\boldsymbol{\theta}), \ldots, \psi_m(\boldsymbol{\theta})$. (Note that an orthogonal set of basis vectors spanning the preconditioned stochastic Krylov subspace can be computed using the stochastic

version of Arnoldi's method as presented by Nair.[19]) Let us first write the zero-order BG condition as

$$\langle\{\boldsymbol{A}(\boldsymbol{\theta})\hat{\boldsymbol{u}}(\boldsymbol{\theta}) - \boldsymbol{f}(\boldsymbol{\eta})\}^*\psi_i(\boldsymbol{\theta})\rangle = 0, \qquad \forall\psi_i(\boldsymbol{\theta}) \in \mathcal{K}_m \tag{34}$$

By the substitution of $\boldsymbol{f}(\boldsymbol{\eta}) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{u}(\boldsymbol{\theta})$, the preceding condition becomes

$$\langle\{\hat{\boldsymbol{u}}(\boldsymbol{\theta}) - \boldsymbol{u}(\boldsymbol{\theta})\}^*\boldsymbol{A}^*(\boldsymbol{\theta})\psi_i(\boldsymbol{\theta})\rangle = 0, \qquad \forall\psi_i(\boldsymbol{\theta}) \in \mathcal{K}_m \tag{35}$$

From the preceding equation, it can be seen that the BG_0 scheme ensures that the difference between the exact and the approximate solution is $\boldsymbol{A}$ orthogonal to the approximating space $\mathcal{K}_m$. Now, by the use of the elementary properties of orthogonal projectors (see Chapter 1 in the book by Saad[20]), it can be shown that Eq. (35) is the necessary and sufficient condition for $\hat{\boldsymbol{u}}(\boldsymbol{\theta})$ to be a minimizer of the error function $\langle\{\hat{\boldsymbol{u}}(\boldsymbol{\theta}) - \boldsymbol{u}(\boldsymbol{\theta})\}^*\boldsymbol{A}\{\hat{\boldsymbol{u}}(\boldsymbol{\theta}) - \boldsymbol{u}(\boldsymbol{\theta})\}\rangle$. This result, which establishes the optimality of the SRBM-BG_0 scheme, can be stated as follows.

Theorem 2: Let $\hat{\boldsymbol{u}}(\boldsymbol{\theta}) = \boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi}$ be a stochastic reduced basis approximation to the solution of $\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{u}(\boldsymbol{\theta}) = \boldsymbol{f}(\boldsymbol{\eta})$, where $\boldsymbol{A}(\boldsymbol{\theta}) \in \mathbb{C}^{n\times n}$ is a random Hermitian positive definite matrix, $\boldsymbol{u}(\boldsymbol{\theta})$ and $\boldsymbol{f}(\boldsymbol{\eta}) \in \mathbb{C}^n$ are random vectors, $\boldsymbol{\Psi}(\boldsymbol{\theta}) \in \mathbb{C}^{n\times m}$ is a matrix of stochastic basis vectors, and $\boldsymbol{\xi} \in \mathbb{C}^m$ is a vector of undetermined coefficients. If the coefficient vector $\boldsymbol{\xi}$ is computed by imposing the condition $\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi}(\boldsymbol{\theta}) - \boldsymbol{f}(\boldsymbol{\eta}) \perp \boldsymbol{\Psi}(\boldsymbol{\theta})$, then the following deterministic error function is minimized:

$$\Delta_A^m = \langle\{\boldsymbol{u}(\boldsymbol{\theta}) - \hat{\boldsymbol{u}}(\boldsymbol{\theta})\}^*\boldsymbol{A}(\boldsymbol{\theta})\{\boldsymbol{u}(\boldsymbol{\theta}) - \hat{\boldsymbol{u}}(\boldsymbol{\theta})\}\rangle \tag{36}$$

where Δ_A^m denotes the $\boldsymbol{A}$ norm of the error.

An important corollary of Theorem 2 is that when the number of basis vectors in the SRBM-BG_0 scheme is increased, the $\boldsymbol{A}$ norm of the error will never increase, that is, $\Delta_A^{i+1} \leq \Delta_A^i$, where Δ_A^{i+1} and Δ_A^i denote the $\boldsymbol{A}$ norm of the error using $i+1$ and i basis vectors, respectively. This implies that the $\boldsymbol{A}$ norm of the error will converge in a mean-square sense. Note that a similar result can be proved for the $\mathcal{L}_2$ norm of the residual for non-Hermitian coefficient matrices, when an oblique stochastic subspace projection scheme is employed to compute the undetermined coefficients. (The reader is referred to Appendix A in Ref. 19 for an overview of the stochastic Petrov–Galerkin scheme.) Even though we have not formally established such results for the exact BG scheme at the time this paper was written, it is conjectured that the $\boldsymbol{A}$ norm of the error will converge in probability for this scheme.

V. Computational Aspects of SRBMs

In summary, the main computational steps in SRBMs involve the 1) decomposition of a deterministic matrix to compute the preconditioner, 2) recursive computation of the basis vectors, 3) stochastic subspace projection to compute the undetermined coefficients in the reduced basis, and 4) statistical postprocessing of the reduced basis representation. It can be readily shown that SRBMs require $\mathcal{O}(p^k q)$ sparse matrix–vector multiplications in addition to one deterministic analysis, where p and q denote the total number of random variables appearing in the coefficient matrix and the RHS, respectively, and k is the number of basis vectors. This implies that the computational cost goes up exponentially with increase in the number of basis vectors. However, as is shown later via numerical studies, two or three basis vectors are generally sufficient to obtain accurate results for a number of cases.

If the tensors in the basis vectors are explicitly computed, then $\mathcal{O}(p^k q)$ n-dimensional vectors are required to be stored in memory. To reduce the memory requirements, the idea presented earlier in Sec. III.B can be effectively used to simplify the basis vectors of order greater than two. The memory requirements and computational cost will consequently reduce to $\mathcal{O}(p^2 q)$ n-dimensional vectors and sparse matrix–vector operations, respectively.

For the exact BG scheme, there is an additional memory requirement of $\mathcal{O}(p^k q)$ scalars if explicit expressions for the random functions describing the coefficients of the reduced basis are desired (see Appendix B). Clearly, this could be prohibitive for systems with a large number of random parameters. As discussed earlier, one way to reduce this additional memory requirement would be to replace all terms of order greater than three in the reduced-order equations by their ensemble averages, which will reduce the additional memory requirement to $\mathcal{O}(p^3 q)$. Furthermore, because most of the computationally intensive steps in SRBMs involve independent sparse matrix–vector operations, this class of algorithms is expected to scale well on parallel computing architectures.

In considering the computation of response statistics, let us first consider the case when the zero-order BG scheme is used for stochastic subspace projection. This leads to a random polynomial for the response process of the form

$$\hat{\boldsymbol{u}}(\boldsymbol{\theta}) = \sum_{i=0}^{m} \xi_i \psi_i(\boldsymbol{\theta}) \tag{37}$$

Because the coefficients ξ_i are deterministic scalars, the mean and covariance of the response can be expressed in terms of the statistics of the basis vectors as follows:

$$\langle\hat{\boldsymbol{u}}(\boldsymbol{\theta})\rangle = \sum_{i=0}^{m} \xi_i \langle\psi_i(\boldsymbol{\theta})\rangle \tag{38}$$

$$\boldsymbol{\Sigma}_{\hat{u}} = \langle\hat{\boldsymbol{u}}(\boldsymbol{\theta})\hat{\boldsymbol{u}}^*(\boldsymbol{\theta})\rangle = \langle\boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi}\boldsymbol{\xi}^*\boldsymbol{\Psi}^*(\boldsymbol{\theta})\rangle$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{m} \xi_i \xi_j^* \langle\psi_i(\boldsymbol{\theta})\psi_j^*(\boldsymbol{\theta})\rangle \tag{39}$$

Note that expectation operations in Eqs. (38) and (39) can be analytically carried out using the joint statistics of θ_i and η_i. (See, for example, McCullagh[21] and Chapter 4 of Ghanem and Spanos[2] for a detailed exposition on statistical analysis of random polynomials.)

The expressions for the mean and covariance matrix suggest that the memory requirements can be significantly reduced if no other statistical quantities are to be computed. To achieve this, we may first compute the coefficients of the stochastic reduced basis using the expressions in Appendix A, without explicitly computing the tensors in the basis vector representation. Subsequently, the mean and covariance of the response can be computed using Eqs. (38) and (39). Clearly, this two-step procedure achieves a significant reduction in the memory requirements at the expense of increased computations. Note that the need for explicitly computing the basis vectors only arises if the response PDF is to be computed in the postprocessing stage by sampling Eq. (37).

In contrast, for the exact BG scheme, the coefficients of the reduced basis turn out to be highly nonlinear functions of the random variables (Appendix B). Hence, analytical solutions for the statistics of $\hat{\boldsymbol{u}}(\boldsymbol{\theta})$ are not readily possible. Fortunately, sampling the reduced basis representation involves only a few operations when $p \ll n$. In particular, the first- and second-order SRBMs require $\mathcal{O}(p^3)$ and $\mathcal{O}(p^5)$ operations, respectively. For problems with a large number of random parameters, the operation count can be reduced by replacing the higher-order terms by their ensemble averages. Hence, Monte Carlo integration techniques can be readily applied to compute the response statistics efficiently.

In the context of reliability analysis (for example, Wu[22]), SRBMs allow the derivation of an explicit expression for the multidimensional limit state curve. This potentially enables the efficient computation of failure probabilities without resorting to first- or second-order reliability approximations. In fact, the complete PDF of the response may be computed by employing simulation schemes and density estimation techniques. Alternatively, the maximum entropy principle[23] could be used to reconstruct the PDF of the solution using the statistical moments computed using Eq. (37).

VI. Demonstration Examples

In this section, we apply SRBMs to some problems in structural mechanics. MCS studies using exact structural analysis are conducted to generate benchmark results, against which the various approximate methods are compared. We present results for the first- and second-order SRBMs, that is, using two and three basis vectors, respectively. When the zero-order BG scheme is used to compute the undetermined coefficients in the reduced basis, the SRBM of

order k is referred to as SRBMk-BG$_0$, where $k+1$ is the number of basis vectors. Similarly, when the exact BG scheme is used, the kth-order SRBM is referred to as SRBMk-BG. We compare the results computed using SRBMs with those obtained using the first- and second-order Neumann expansion scheme, referred to as NEU1 and NEU2, respectively. For examples 1 and 2, we also present results for the first-order polynomial chaos scheme,[2] referred to as PC1.

A. Example 1: Static Structural Analysis

The first example considered is a 10-bar truss structure (Fig. 1) taken from Penmetsa et al.,[24] where an approach based on Fourier transforms was compared with various reliability analysis techniques. In this problem, the cross-sectional areas of all of the truss members are modeled as uncorrelated Gaussian random variables with mean 2.5 and standard deviation (STD) σ_θ. The value of σ_θ is set at 0.3 and 0.5 for cases 1 and 2, respectively. For each case, MCS using exact static analysis with a sample size of 300,000 is used to generate benchmark results (henceforth referred to as exact MCS). The results computed using SRBMs, the Neumann series, and the first-order polynomial chaos are compared with these benchmark results.

The average percentage errors in the mean and STD across all of the DOF are shown in Figs. 2 and 3 for cases 1 and 2, respectively. It can be seen that the Neumann series converge very slowly to the benchmark results, particularly for the STD of the displacement. In contrast, SRBMs converge more rapidly to the exact results when the number of basis vectors is increased. PC1 generally tends to give results that are comparable to NEU2. Note that, for this particular problem (with eight DOF and 10 random variables), application of PC1 leads to the requirement of solving an 88×88 deterministic system of equations. Given this, it is remarkable that SRBM1-BG$_0$ gives results of similar accuracy by solving a 2×2 deterministic system of equations. Also note that SRBM2-BG gives exact results for the first two statistical moments for both cases. In summary, the results support the theoretical evidence for the optimality of the preconditioned stochastic Krylov subspace presented earlier.

Note that the size of the deterministic system of equations to be solved for the undetermined coefficients grows linearly for SRBMs. In contrast, the growth in dimensionality for the polynomial chaos scheme is significantly higher; for example, the second-order polynomial chaos scheme results in a 528×528 system of equations for the simple example problem considered here.

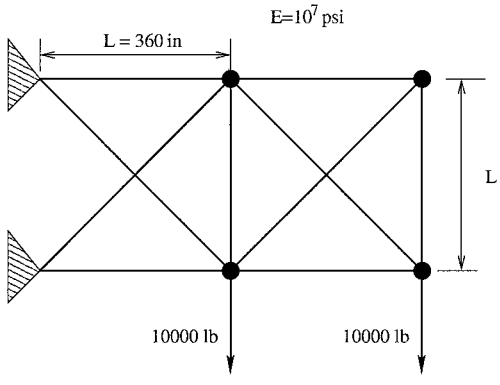

Fig. 1 Truss structure, 10 bars.

B. Example 2: Structural Reliability Analysis

In this section, we present results for reliability analysis of the 10-bar truss structure considered in example 1. As in Ref. 24, the cross-sectional areas of all of the truss members are modeled as uncorrelated Gaussian random variables, with mean 2.5 and STD 0.5. The reliability analysis problem involves computing the multidimensional integral

$$P_f = \int_{g(a) > 1.8} f_a(\boldsymbol{a})\, \mathrm{d}\boldsymbol{a} \tag{40}$$

where P_f denotes the probability of structural failure, $\boldsymbol{a} \in \mathbb{R}^{10}$ denotes the vector of random cross-sectional areas and $f_a(\boldsymbol{a})$ denotes its joint PDF, and $g(\boldsymbol{a})$ is the limit state function that is defined for this problem as $g(\boldsymbol{a}) = \max\{\boldsymbol{u}(\boldsymbol{\theta})\}$, where $\boldsymbol{u}(\boldsymbol{\theta}) \in \mathbb{R}^8$ denotes the random displacement vector. Clearly, it would be preferable to simplify SRBMs to compute P_f analytically. However, an analytical expression for P_f can be readily derived only when SRBM1-BG$_0$ is used. In the present study, we use a simulation scheme to compute P_f because SRBMs provide an explicit expression for the displacement vector.

We compare SRBMs with various approximate reliability analysis methods, the performances of which were studied in Ref. 24.

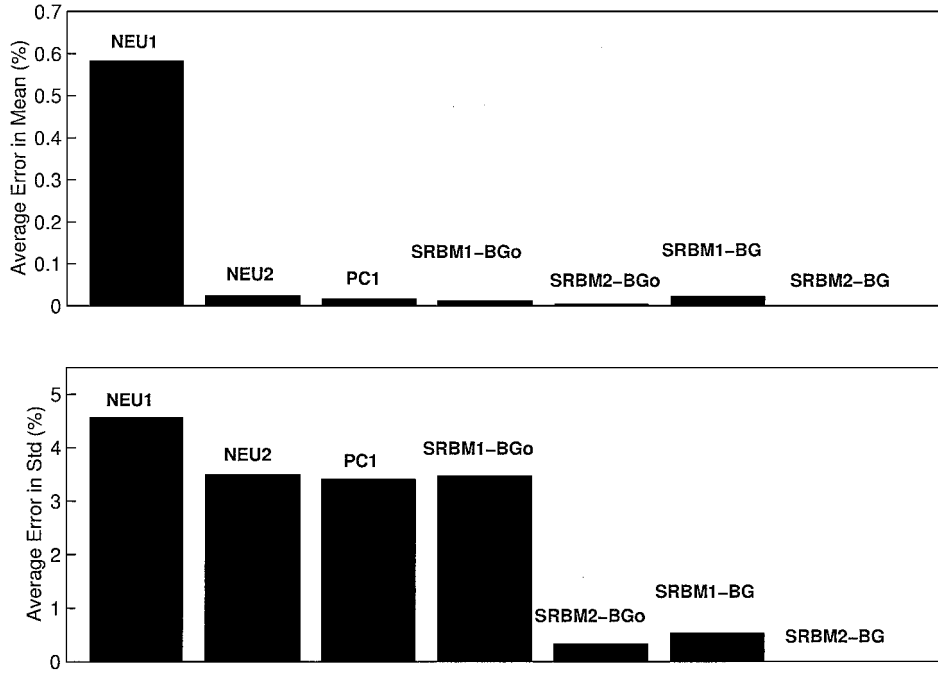

Fig. 2 Average errors in mean and STD of displacement for example 1, case 1, $\sigma_\theta = 0.3$.

We generated benchmark results using a sample size of 300,000 to compute the probability of failure using exact static analysis as well as the SRBMs (except for SRBM1-BG_0 and PC1, where P_f can be computed analytically). The percentage errors in P_f computed using the first- and second-order SRBMs are compared with other techniques in Table 1. We observed that our benchmark result (MCS1) is around 1.8% lower than that reported in Ref. 24 (referred to as MCS2 in Table 1). Hence, we also show in parentheses the percentage error of all the methods vis-á-vis MCS2.

Note that except for the SRBMs and PC1, all other techniques compute as a first step the most probable point (MPP) of failure using an iterative scheme, and then construct an approximate model around it. In contrast, the SRBMs circumvent the MPP computation step by constructing a global model using a set of stochastic basis vectors. In spite of this, it can be readily seen from Table 1 that the SRBMs give more accurate results as compared to the other state-of-the-art reliability analysis techniques considered here. Also note that SRBM1-BG_0 (which is a first-order method) gives better results than the FORM and the second-order reliability methods, referred to as BREITUNG, TVEDT2, and TVEDT3. Clearly, the accuracy of the SRBMs could have been improved even further by using a preconditioning matrix at the MPP (i.e., the inverse of the stiffness matrix computed at this point) for computing the basis vectors. Further work is also required to introduce simplifying assumptions for analytically computing P_f.

Table 1 Comparison of methods for reliability analysis of a 10-bar truss structure

Method	P_f	Error, %
MCS1	0.1384	——
SRBM1-BG	0.1358	−1.88 (−3.76)
SRBM2-BG	0.1383	0.0 (−1.94)
SRBM1-BG_0	0.1150	−16.9 (−18.8)
SRBM2-BG_0	0.1399	2.74 (0.85)
PC1	0.1170	−15.44 (−17.1)
MCS2[a]	0.1411	——
FFT[a]	0.1326	−4.19 (−6.04)
FORM[a]	0.0862	−37.72 (−38.90)
BREITUNG[a]	0.0907	−34.47 (−35.71)
TVEDT2[a]	0.0951	−31.29 (−32.61)
TVEDT3[a]	0.0966	−30.20 (−31.53)

[a]Results for these methods have been taken from Penmetsa et al.[24]

C. Example 3: Frequency Response Analysis

In this section, we present results for frequency response analysis of a cantilevered network of 20 Euler–Bernoulli beams (Fig. 4), each with random Young's modulus and mass density. The structure is modeled using four elements for each beam member, which leads to a finite element model with 210 DOF. The axial and flexural rigidity of each structural member are modeled as $E_oA(1+\theta_i)$ and $E_oI(1+\theta_i)$, $i=1, 2, \ldots, 20$, respectively. The mass density of each member is modeled as $\rho=\rho_o(1+\theta_i)$, $i=21, 22, \ldots, 40$. Note that θ_i, $i=1, 2, \ldots, 40$, are considered as uncorrelated zero-mean Gaussian random variables with STD 0.05, whereas $E_oA=6.987\times10^6$ N, $E_oI=1.286\times10^3$ Nm^2, and $\rho_o=2.74$ kg/m.

As shown in Fig. 4, the structure is subjected to transverse harmonic excitation at joint 1. The transverse component of the displacement response at joint number 9 is studied in the region of 0–500 Hz at 150 equally spaced points. Results are computed for

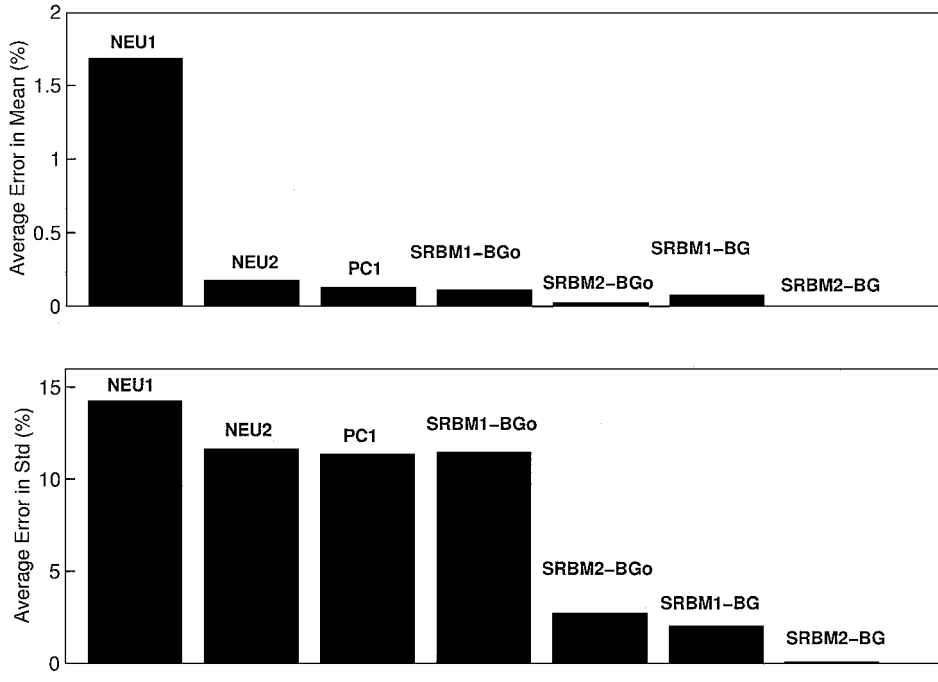

Fig. 3 Average errors in mean and STD of displacement for example 1, case 2, $\sigma_\theta=0.5$.

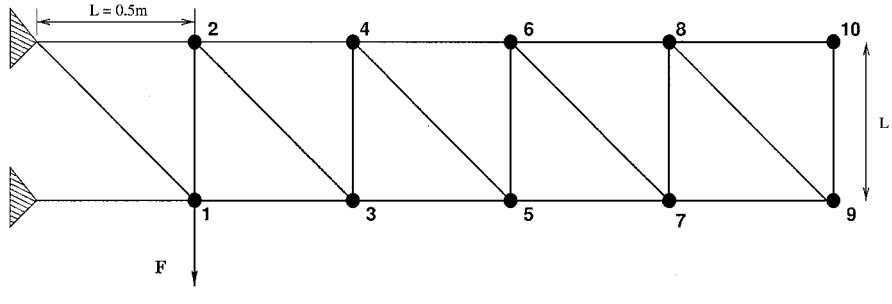

Fig. 4 Cantilevered network of 20 Euler–Bernoulli beams.

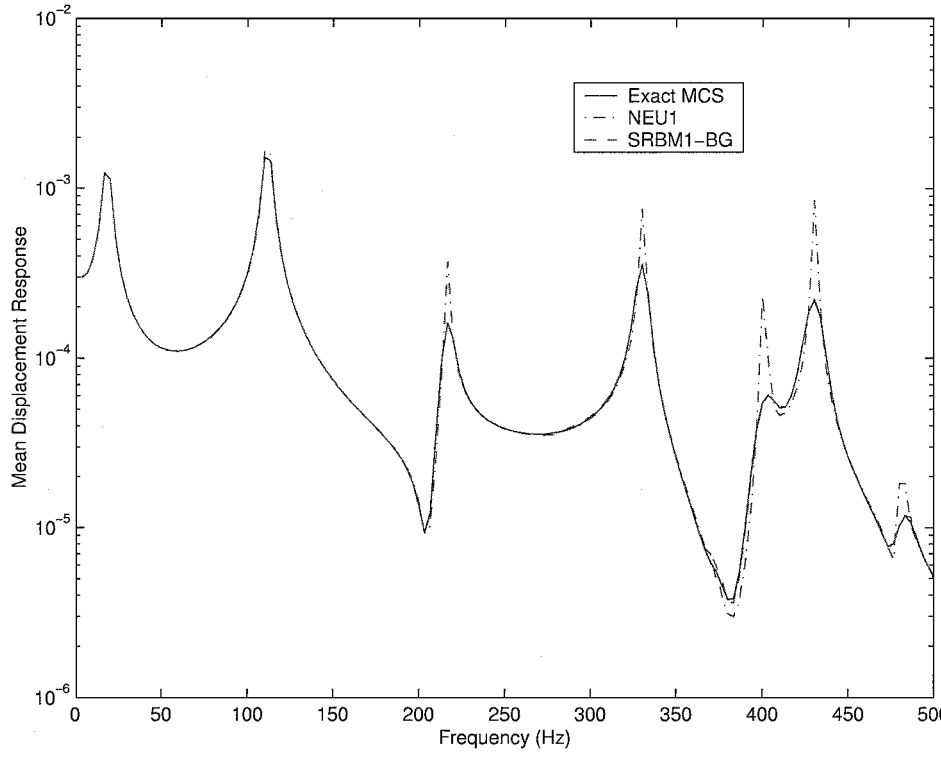

Fig. 5 Comparison of mean displacement response computed using the first-order methods for example 3.

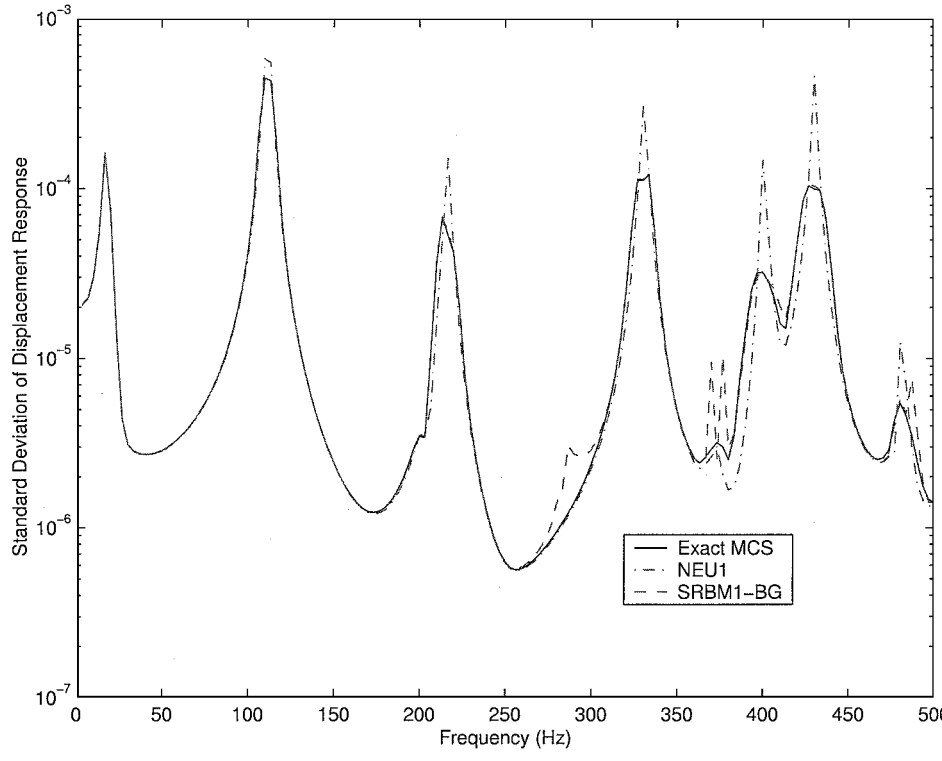

Fig. 6 Comparison of STD of displacement response computed using the first-order methods for example 3.

SRBM1-BG, SRBM2-BG, NEU1, NEU2, and exact MCS using a sample size of 10,000.

The mean and STD of the displacement response computed using the approximate methods are compared with benchmark results generated using MCS and exact structural dynamic analysis in Figs. 5–8. It can be clearly seen that SRBMs give significantly better approximations than the Neumann series. In particular, SRBM2-BG gives accurate approximations for both statistical moments of the displacement response. The accuracy of SRBMs for this problem is remarkable considering that Theorem 2 does not strictly hold for complex symmetric matrices. It is expected that the accuracy of SRBMs can be improved by employing an oblique stochastic projection scheme, which, as mentioned in Sec. IV.C, can be shown to be convergent for non-Hermitian matrices.

It can be observed that NEU2 gives more erroneous results than NEU1 for both statistical moments of the frequency response, particularly near resonance frequencies. This indicates that the Neumann series diverges for this problem. However, the preconditioned

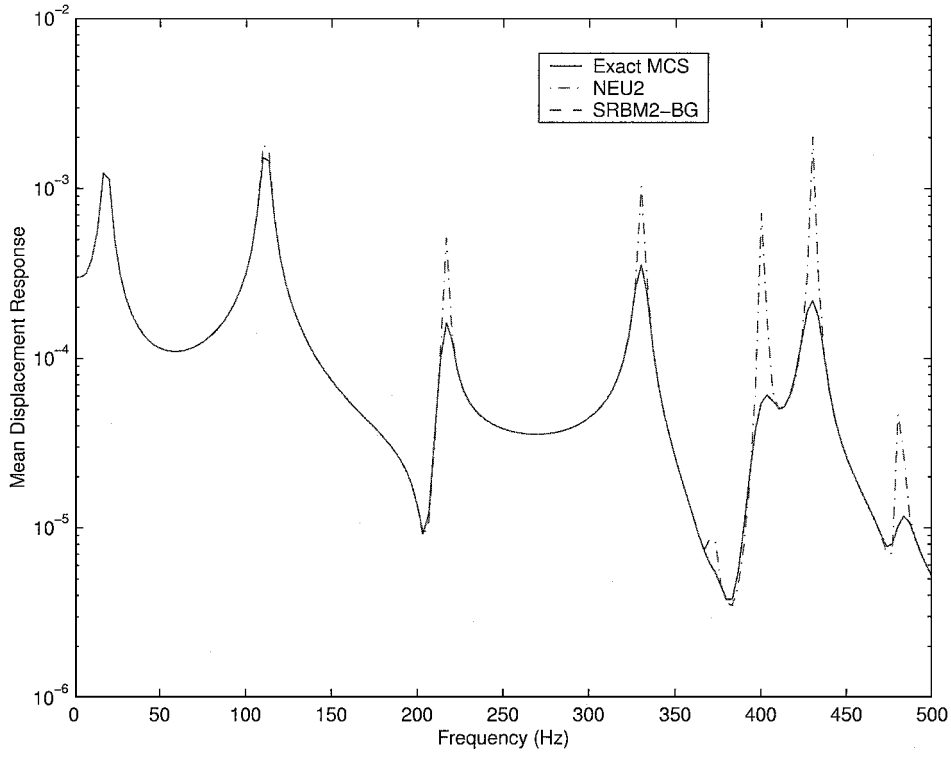

Fig. 7 Comparison of mean displacement response computed using the second-order methods for example 3.

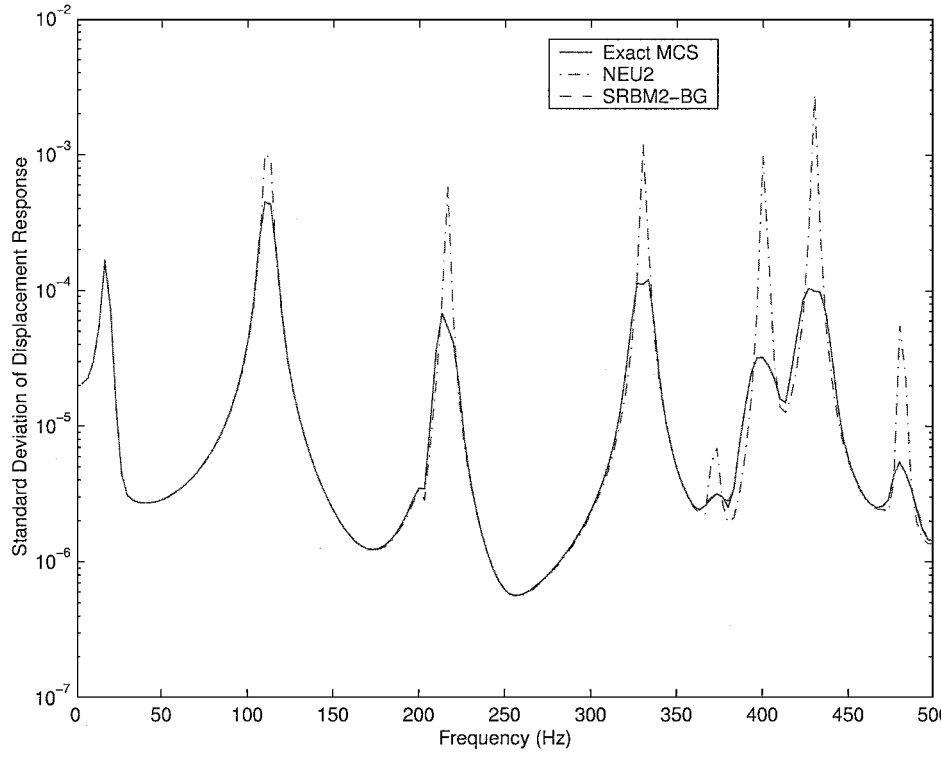

Fig. 8 Comparison of STD of displacement response computed using the second-order methods for example 3.

stochastic Krylov subspace (or the terms of the Neumann series) do provide a rich set of basis vectors for approximating the response process. Hence, using this set of basis vectors, the response statistical moments computed using the SRBMs are up to orders of magnitude more accurate than those derived from the Neumann series.

VII. Summary

In this paper, we have presented SRBMs for solving linear random algebraic systems of equations, such as those arising from discretization of linear stochastic PDEs in space, time, and the random dimension. We introduced the idea of representing the solution vector as a linear combination of stochastic basis vectors with undetermined coefficients. A theoretical justification for employing basis vectors spanning the preconditioned stochastic Krylov subspace was outlined. Subsequently, we presented stochastic variants of the BG scheme for computing the undetermined coefficients in the reduced basis. It is shown that the $\boldsymbol{A}$ norm of the error converges for these projection schemes, when the number of basis vectors is increased.

We also closely examined the computational aspects of SRBMs. In particular, an efficient scheme was presented for recursively computing basis vectors spanning the preconditioned stochastic Krylov subspace for static and dynamic structural analysis. Some ideas for reducing the computational cost and memory requirements of SRBMs were also discussed. Similar to the polynomial chaos projection scheme,[2] a major advantage of SRBMs is that an explicit representation of the response quantities in terms of the random system parameters can be achieved. As a result, the response statistics can be efficiently computed in the postprocessing stage. However, because SRBMs lead to a reduced-order system of equations, our approach is expected to be computationally more efficient than the polynomial chaos projection scheme, which leads to a system of equations with increased dimensionality.

Numerical studies were presented for static and dynamic analysis of randomly parameterized structural systems. The results obtained using SRBMs have been compared with the Neumann series and other relevant techniques in the literature. We demonstrate that SRBMs converge much faster to the exact statistics than the Neumann series. For some of the cases, the response statistical moments computed using SRBMs are up to 10 times more accurate compared to the Neumann series. The results also indicate that SRBMs can provide accuracy that is comparable to, or better than, the polynomial chaos scheme at a much lower computational cost. This suggests that SRBMs can be employed as an efficient solver in the SSFEM scheme of Ghanem and Spanos.[2]

The present research has important ramifications from a theoretical point of view. From the theoretical perspective, we have highlighted how existing results in numerical linear algebra on Krylov subspace methods[24] for deterministic systems can be leveraged to devise SRBMs. In numerical linear algebra, methods based on the Krylov subspace have a rich history spanning more than 50 years, and they continue to be an area of extensive research. Hence, the connections made in this paper provide us access to a sound theoretical foundation from which new numerical methods for stochastic problems may be developed.

As computational experience accumulates on a variety of problem domains, we hope to gain a deeper insight into the theoretical and computational properties of SRBMs. This is expected to aid us in the task of devising alternative choices of preconditioners or basis vectors, which may lead to significant reductions in the computational cost and memory requirements. It also remains to be seen whether employing an oblique stochastic projection scheme[19] will improve the accuracy of the results for stochastic structural dynamics. Ideas on extending SRBMs to nonlinear stochastic systems, algebraic random eigenvalue problems,[19] and a posteriori error estimation schemes[19] merit further investigation. It is hoped that the formulations presented in this paper will accelerate the development of efficient stochastic subspace projection schemes for tackling a wider class of problems in stochastic mechanics.

Appendix A: Reduced-Order Terms in SRBM–BG$_0$ Scheme

The expressions for the deterministic reduced-order matrices which arise in the kth-order SRBM using the zero-order BG scheme [see Eq. (29)] are summarized hereafter. $\langle \boldsymbol{L}_R(\boldsymbol{\theta})\rangle \in \mathbb{C}^{(m+1)\times(m+1)}$ is a deterministic matrix, a typical element of which can be computed as

$$\begin{aligned}\langle \boldsymbol{L}_R(k,l)\rangle &= \left\langle \psi_k^*(\boldsymbol{\theta})\boldsymbol{L}\psi_l(\boldsymbol{\theta})\right\rangle \\ &= \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_k}\theta_{j_1}\theta_{j_2},\ldots,\theta_{j_l}\right\rangle \left(\boldsymbol{b}^k_{i_1 i_2,\ldots,i_k}\right)^* \boldsymbol{L}\left(\boldsymbol{b}^l_{j_1 j_2,\ldots,j_l}\right) \\ &+ \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_k}\theta_{j_1}\theta_{j_2},\ldots,\theta_{j_l}\eta_{j_{l+1}}\right\rangle \left(\boldsymbol{b}^{k1}_{i_1 i_2,\ldots,i_k}\right)^* \boldsymbol{L}\left(\boldsymbol{c}^l_{j_1 j_2,\ldots,j_{l+1}}\right) \\ &+ \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_k}\eta_{i_{k+1}}\theta_{j_1}\theta_{j_2},\ldots,\theta_{j_l}\right\rangle \left(\boldsymbol{c}^k_{i_1 i_2,\ldots,i_{k+1}}\right)^* \boldsymbol{L}\left(\boldsymbol{b}^l_{j_1 j_2,\ldots,j_l}\right) \\ &+ \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_k}\eta_{i_{k+1}}\theta_{j_1}\theta_{j_2},\ldots,\theta_{j_l}\eta_{j_{l+1}}\right\rangle \\ &\times \left(\boldsymbol{c}^k_{i_1 i_2,\ldots,i_{k+1}}\right)^* \boldsymbol{L}\left(\boldsymbol{c}^l_{j_1 j_2,\ldots,j_{l+1}}\right)\end{aligned}$$

$\langle \boldsymbol{\Pi}_R(\boldsymbol{\theta})\rangle \in \mathbb{C}^{(m+1)\times(m+1)}$ is a deterministic matrix that can be computed as

$$\begin{aligned}\langle \boldsymbol{\Pi}_R(k,l)\rangle &= \sum_{i=1}^{p}\left\langle \theta_i \psi_k^*(\boldsymbol{\theta})\boldsymbol{\Pi}_i\psi_l(\boldsymbol{\theta})\right\rangle \\ &= \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_{k+1}}\theta_{j_1}\theta_{j_2},\ldots,\theta_{j_l}\right\rangle \left(\boldsymbol{b}^k_{i_1 i_2,\ldots,i_k}\right)^* \boldsymbol{\Pi}_{k+1}\left(\boldsymbol{b}^l_{j_1 j_2,\ldots,j_l}\right) \\ &+ \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_{k+1}}\theta_{j_1}\theta_{j_2},\ldots,\theta_{j_l}\eta_{j_{l+1}}\right\rangle \left(\boldsymbol{b}^k_{i_1 i_2,\ldots,i_k}\right)^* \\ &\times \boldsymbol{\Pi}_{k+1}\left(\boldsymbol{c}^l_{j_1 j_2,\ldots,j_{l+1}}\right) \\ &+ \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_{k+1}}\eta_{i_{k+2}}\theta_{j_1}\theta_{j_2},\ldots,\theta_{j_l}\right\rangle \left(\boldsymbol{c}^k_{i_1 i_2,\ldots,i_{k+1}}\right)^* \\ &\times \boldsymbol{\Pi}_{k+1}\left(\boldsymbol{b}^l_{j_1 j_2,\ldots,j_l}\right) \\ &+ \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_{k+1}}\eta_{i_{k+2}}\theta_{j_1}\theta_{j_2},\ldots,\theta_{j_l}\eta_{j_{l+1}}\right\rangle \left(\boldsymbol{c}^k_{i_1 i_2,\ldots,i_{k+1}}\right)^* \\ &\times \boldsymbol{\Pi}_{k+1}\left(\boldsymbol{c}^l_{j_1 j_2,\ldots,j_{l+1}}\right)\end{aligned}$$

Similarly, a typical element of $\langle \tilde{\boldsymbol{f}}(\boldsymbol{\theta},\boldsymbol{\eta})\rangle \in \mathbb{C}^{m+1}$ can be evaluated as

$$\begin{aligned}\langle \tilde{\boldsymbol{f}}(k)\rangle &= \left\langle \psi_k^*\right\rangle \boldsymbol{f}_o + \sum_{i=1}^{q}\left\langle \psi_k^*\eta_i\right\rangle \boldsymbol{f}_i \\ &= \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_k}\right\rangle \left(\boldsymbol{b}^k_{i_1 i_2,\ldots,i_k}\right)^* \boldsymbol{f}_o \\ &+ \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_k}\eta_{i_{k+1}}\right\rangle \left(\boldsymbol{c}^k_{i_1 i_2,\ldots,i_{k+1}}\right)^* \boldsymbol{f}_o \\ &+ \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_k}\eta_{k+1}\right\rangle \left(\boldsymbol{b}^k_{i_1 i_2,\ldots,i_k}\right)^* \boldsymbol{f}_{i_{k+1}} \\ &+ \left\langle \theta_{i_1}\theta_{i_2},\ldots,\theta_{i_k}\eta_{i_{k+1}}\eta_{i_{k+2}}\right\rangle \left(\boldsymbol{c}^k_{i_1 i_2,\ldots,i_{k+1}}\right)^* \boldsymbol{f}_{i_{k+2}}\end{aligned}$$

In the notation used here, repeated indices indicate summation with respect to that index over its range. Given the joint statistics of θ_i and η_i, the expectation operations appearing in the preceding equations can be readily carried out. Hence, the undetermined coefficients in the stochastic reduced basis can be computed as $\boldsymbol{\xi} = [\langle \boldsymbol{L}_R\rangle + \langle \boldsymbol{\Pi}_R\rangle]^{-1}\langle \tilde{\boldsymbol{f}}\rangle$. Note that the expressions for the reduced-order matrices can be greatly simplified if the vectors $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are modeled as zero-mean uncorrelated Gaussian random variables. In fact, such a simplification was used to develop a library of subroutines for the numerical studies presented here. An even greater simplification results if the interaction terms in the basis vectors of order greater than two are assumed to be zero (see Sec. III.B).

Appendix B: Reduced-Order Terms in SRBM–BG Scheme

For the sake of notational simplicity, consider the case when the RHS in Eq. (4) is deterministic. Then the reduced-order coefficient matrix and the RHS for the first-order SRBM, which uses the exact BG scheme [see Eq. (33)], can be written using repeated index notation as

$$\boldsymbol{L}_R(\boldsymbol{\theta}) + \boldsymbol{\Pi}_R(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{t}_o^*\boldsymbol{L}\boldsymbol{t}_o + \theta_i a_i & \theta_i b_i + \theta_i\theta_j C_{ij} \\ \theta_i\tilde{b}_i + \theta_i\theta_j\tilde{C}_{ij} & \theta_i\theta_j D_{ij} + \theta_i\theta_j\theta_k E_{ijk}\end{bmatrix} \tag{B1}$$

$$\tilde{\boldsymbol{f}}(\boldsymbol{\theta},\boldsymbol{\eta}) = \begin{bmatrix} \boldsymbol{t}_o^* & \boldsymbol{f}_o \\ \theta_i & f_i\end{bmatrix} \tag{B2}$$

where $\boldsymbol{t}_o = \boldsymbol{L}^{-1}\boldsymbol{f}_o \in \mathbb{C}^n$, $a_i = \boldsymbol{t}_o^*\boldsymbol{\Pi}_i\boldsymbol{t}_o$, $b_i = \boldsymbol{t}_o^*\boldsymbol{L}\boldsymbol{b}_i^1$, $C_{ij} = \boldsymbol{t}_o^*\boldsymbol{\Pi}_i\boldsymbol{b}_j^1$, $\tilde{b}i = (\boldsymbol{b}_i^1)^*\boldsymbol{L}\boldsymbol{t}_o$, $\tilde{C}_{ij} = (\boldsymbol{b}_j^1)^*\boldsymbol{\Pi}_i\boldsymbol{t}_o$, $D_{ij} = (\boldsymbol{b}_i^1)^*\boldsymbol{L}\boldsymbol{b}_j^1$, $E_{ijk} = (\boldsymbol{b}_i^1)^*\boldsymbol{\Pi}_j\boldsymbol{b}_k^1$, and $f_i = (\boldsymbol{b}_i^1)^*\boldsymbol{f}_o$.

By the use of Eqs. (B1) and (B2), an explicit expression can be derived for the random functions in the reduced basis representation. Similar expressions can also be derived when three basis vectors are used in the reduced basis.

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